Seminar on Galois Groups and Fundamental Groups

Talk 4
Covering Theory

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Abstract

Covering Spaces constitute an important contribution in understanding the homotopy theory and Riemanninan geometry among other fields. The far reach of this theory is due to the visual flavour imbibed in it and its ability to commute to other areas of study. The close connection between the Galois theory of Field extensions and the Fundamental groups helps in gaining a lot of insight about more complicated fields and spaces, be it the cyclotomic fields or some weird topological spaces. An important practical application of covering spaces occurs in charts on $SO(3)$, the rotation group. This group occurs widely in engineering, due to 3-dimensional rotations being heavily used in navigation and nautical engineering. In what follows, we shall introduce the notion of covering spaces and some of its interesting consequences.

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1 Motivation

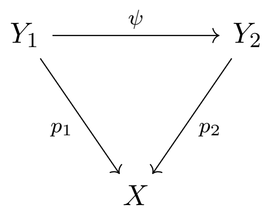
The Galois theory of Fields is vastly studied for its topological analog that has a very nice visual and computational advantage. The ideas stems as follows: Let L/K be any finite separable field extension, then we can see the base field K as a single point upon which the extension field is set out as a finite set of (discrete) points that maps directly to this base point. We shall take care that the continuity is preserved in such a process and hence the Galois theory equips this situation with the notion of *Absolute Galois Group* that leaves the base point fixed.

But more information could be gained if we consider the base field as a space rather than a single point and hence the notion of *space over space* can be introduced homologously. The role of field extensions would then be played by certain continuous surjections, called *covers*, whose fibres are finite (or, even more generally, arbitrary discrete) spaces. Under some restrictions on the base space one can also develop a topological analogue of the Galois theory of fields with the part of the absolute Galois group being taken by the fundamental group of the base space. After one has equipped this situation very pragmatically and carefully, a lot of suprising analogues between topological spaces and Fields can be unravelled, thus contributing to the richness of both fields.

2 The Theory of Covering Spaces

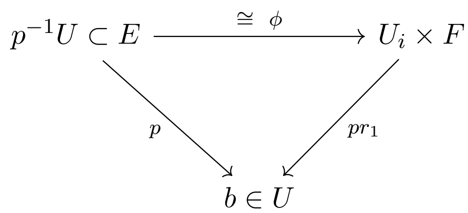
Definition 2.1 (Space over space)**.** Let X be any topological space. A *space over a space* X is any topological space Y (allowing the case $Y = X$) along with a continuous map $p : Y \to X$. With morphism as explained below, this forms a category TOP_B which is called the *space over* B.

Suppose if Y_1, Y_2 are two such spaces over X with $p_1 : Y_1 \to X$ and $p_2 : Y_2 \to X$, then a morphism $\psi : Y_1 \to Y_2$ is such that the following diagram commutes.



Definition 2.2 (Fibre Bundle)**.** Let B be any topological space. A *fibre bundle* E *over* B is a topological space E with fibres F equipped with a continuous map $p : E \to B$. Such a space should satisfy the local trivality condition as follows,

Local Trivialisation: Let $p : E \to B$ be continuous map and let $U \subset B$ be open and one can assume the map p is surjective to avoid empty fibres. A *trivialisation* of U w.r.t. p is a homeomorphism $\phi : p^{-1}U \to U \times F$ such that $pr_1 \circ \phi = p$ (*i.e.*) the following triangle commutes. This condition determines F upto homeomorphism, since ϕ induces a homeomorphism $p^{-1}U$ with $\{u\} \times F$.



Local Triviality of the maps: We say the map p is *locally trivial* if for each $b \in B$, there exists an open neighbourhood $U \subset B$ such that U is a local trivialisation. We say the map p is *trivial* if the set B itself has a local trivialisaton.

As a simple consequence, if p is trivial, then it is locally trivial, but not the converse. We shall also see an interesting example obstructing the converse.

The triple (E, F, p) is sometimes called the fibre bundle over B.

Definition 2.3 (Cover)**.** A *Cover* or more generally a Covering Space is a special fibre bundle with fibres in Set. In other words, a covering has a discrete fibres. Here, we use both cover and covering space interchangeably.

Remark 2.4. The collection of all fibre bundles (resp. covers) over B form a category called the Fib_B (resp. COV_B) whose objects are fibre bundles (resp. covering spaces). The morphism ψ in 2.1 is also called the *bundle map* or *covering map* in the category of Fib_B or COV_B respectively. However, it is important to note that COV_B form a full subcategory of TOP_B.

Example 2.5.

1. Any space can be realised as a covering over itself. We can always take $E = B$ in the definition above. More generally, for any non-empty discrete topological space I, one can take $E = B \times I$ and verify that this is a *trivial* bundle over B with fibres I. These are also called *Trivial Covers.*

2. The One-dimensional bundles over a space are sometimes called a *Line bundle.* The Möbius strip is a *non-trivial* line bundle over $\mathbb{S}^1$ with fibres $I = [0, 1]$.

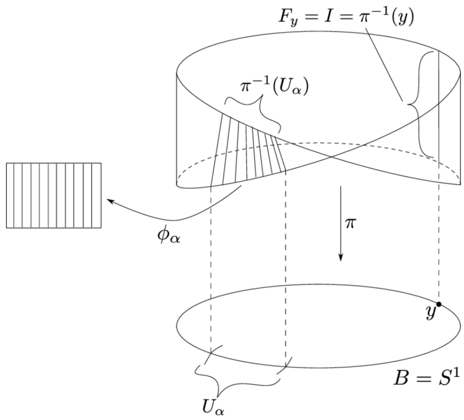


Figure 1: Möbius band as a line bundle over $\mathbb{S}^1$ [**4**]

It has a circle that runs lengthwise along the center of the strip as a base B and a line segment for the fiber F, so the Möbius strip is a bundle of the line segment over the circle $\mathbb{S}^1$. A neighborhood U of $\pi(x) \in B$ (where $x \in E$) is an arc here. A homeomorphism exists that maps the preimage of U (the trivializing neighborhood) to a slice of a cylinder which is curved, but not twisted! This pair locally trivialises the strip. The corresponding trivial bundle $B \times F$ would be a cylinder, but the Möbius strip has an overall "twist". This twist is visible only globally; locally the Möbius strip and the cylinder are identical and indistinguishable.

3. The space $\mathbb{R}$ is the cover of S^1 defined by the exponential map $p : \mathbb{R} \to \mathbb{S}^1$, $t \mapsto e^{2\pi i t}$. The fibres of this bundle can be found to be $\mathbb{Z}$. One can easily check that it is is not a trivial bundle as $\mathbb{R} \ncong \mathbb{S}^1 \times \mathbb{Z}$.

4. The tangent bundle $T\mathbb{S}^1$ over $\mathbb{S}^1$ is *trivial* with fibre as $\mathbb{R}$. The map $F : \mathbb{S}^1 \times \mathbb{R} \to T\mathbb{S}^1 \subset \mathbb{R}^2 \times \mathbb{R}^2$ given by $((x, y), t) \mapsto ((x, y), t(-y, x))$ However, by the *Hairy-Ball theorem* we

have that TS^2 is *non-trivial* bundle over $\mathbb{S}^2$. In fact, the bundle $T\mathbb{S}^n$ over $\mathbb{S}^n$ is trivial $\iff n = 1, 3, 7$.

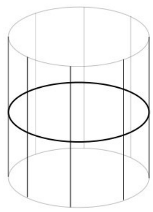


Figure 2: Tangent bundle over $\mathbb{S}^1$

The vertical lines represent the tangent spaces attached disjointly to $\mathbb{S}^1$ represented by the black circle.

5. In fact, if M is any smooth n-manifold, then every tangent bundle $TM \to M$ is a locally trivial bundle with fibres $\mathbb{R}^n$. A manifold whose tangent bundle is isomorphic to a trivial bundle is called *Parallelizable.*

Proposition 2.6 (Every cover is locally a trivial cover)**.** *A space E over B is a cover if and only if each point of B has an open neighbourhood V such that the restriction of the projection $p : E \to B$ to $p^{-1}V$ is isomorphic (as a space over V) to a trivial cover.*

Proof. $\Longrightarrow$: This fact trivially follows from the example 1 as provided above.
$\Longleftarrow$: If V is any open neighbourhood in the base space B with a decomposition $p^{-1}V \cong \bigsqcup_{i\in I} U_i$ for some index set I, mapping $u_i \in U_i$ to the pair $(p(u_i), i)$ defines a homeomorphism of $\bigsqcup_{i\in I} U_i$ onto $V \times I$, where I is endowed with the discrete topology. By construction this is an isomorphism of covers of V. □

Corollary 2.7. *If B is connected, the fibres of p are all homeomorphic to the same discrete space I.*

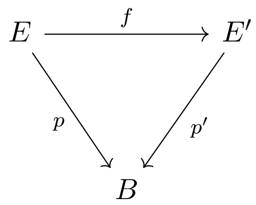
Remark 2.8 (Properties of Covering Spaces)**.**

1. **The product of coverings is again a covering.** If $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ are two covering, then the map $p_1 \times p_2 : E_1 \times E_2 \to B_1 \times B_2$ is again a covering of $B_1 \times B_2$. For example: $\mathbb{R} \times \mathbb{R}$ is the covering of $\mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}^2$ with fibres $\mathbb{Z} \times \mathbb{Z}$.

2. **Covering are preserved under pullbacks.** Consider the following pullback square. If p' is a covering, then so is p.

$$\begin{array}{ccc} E & \longrightarrow & E' \\ \big\downarrow{\scriptstyle p} & \lrcorner & \big\downarrow{\scriptstyle p'} \\ B & \longrightarrow & B' \end{array}$$

3. Let B be any locally connected topological space. If the p and p' are covering in the following commutative square,



then f is a covering.

Definition 2.9 (Even action)**.** Let G be a group acting continuously from the left on a topological space X . The action of G is *even* if each point $x \in X$ has some open neighbourhood U such that the open sets gU (also called orbits) are pairwise disjoint for all $g \in G$. We represent such an action by $G\backslash X$.

Lemma 2.10. *If G is a group acting evenly on a connected space X, then the projection $pr_G : X \to G\backslash X$ turns X into a cover of $G\backslash X$.*

Proof. Firstly, note that this projection mapping $pr_X : X \to G\backslash X$ is surjective. Now since the action is even, every point $x \in X$ has some open neighbourhood U such that gU is pairwise disjoint in $G\backslash X$. Now taking back these to the space X by pr_X^{-1}, these sets are mapped to their disjoint representatives and the claim follows naturally. □

Remark 2.11. The covering defined above is also called the *Projection covering.* The group action is also sometimes called the *Properly Discontinuous action.* The *even* action of group is equivalent to claiming that the group acts both properly and freely on the space X.

Example 2.12 (Even actions form nice spaces)**.**

1. The translation action of $\mathbb{Z}$ on $\mathbb{R}$ defined by $\mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ such that $(x, n) \mapsto x + n$, for $n \in \mathbb{Z}$. Alternatively, these can be viewed as automorphisms of the space $\mathbb{R}$ given by the translation of points. Now by the lemma above, $\mathbb{R}$ is immediately seen to be the cover of the space $\mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$ as it is an even action.

2. More generally one can take this to arbitrary dimensions and form the circle group $\mathbb{S}^n \cong \mathbb{R}^n/\mathbb{Z}^n$. The map is defined as follows: Take any basis $\{x_1, \dots x_n\}$ of the vector space $\mathbb{R}^n$ and make $\mathbb{Z}^n$ act on $\mathbb{R}^n$ so that the i-th direct factor of $\mathbb{Z}^n$ acts by translation by x_i . This action is clearly even and turns $\mathbb{R}^n$ into a cover. The resulting quotient space is called the *linear torus.* When $n = 2$ we get the usual torus. The subgroup Λ of $\mathbb{R}^n$ generated by the x_i is usually called a *lattice.* Thus linear tori are quotients of $\mathbb{R}^n$ by lattices.

3. For an integer $n > 1$ denote by μ_n the group of n-th roots of unity. We can multiply by elements of μ_n and thus it defines an even action on $\mathbb{C}^* := \mathbb{C}\backslash\{0\}$. The multiplicative group $\mathbb{C}^*$ covers the space $\mathbb{C}^*/\mu_n$ defined by the map $p_n\mathbb{C}^* \to \mathbb{C}^*/\mu_n$. In fact, the map $z \mapsto z^n$ defines natural homeomorphism of $\mathbb{C}^*/\mu_n$ onto $\mathbb{C}^*$ and via this homeomorphism p_n becomes identified with the cover $\mathbb{C}^* \to \mathbb{C}^*$ given by $z \mapsto z^n$.

3 Some Interesting Consequences

1. The calculation of fundamental group of certain spaces can be simplifed. In particular, if the space X is simply connected with an even action by group $G\backslash X$, then $\pi_1(G\backslash X) \cong G$. The result uses the fact that the map $X \twoheadrightarrow G\backslash X$ is a covering.

For example, by setting $X = \mathbb{S}^2$ and $G = \mathbb{Z}/2$, and let G act on X by taking $x \mapsto (-x)$. This action identifies the anti-podal points on the unit sphere and further this action can be easily found to be even. The space obtained by this corresponding action is the Real projective Space RP^2 and hence by the above fact we have that $\pi_1(G\backslash X) = \pi_1(RP^2) = \mathbb{Z}/2$. More generally, for all $n \geq 2$, $\pi_1(RP^n) = \mathbb{Z}_2$ holds true.

2. **Triviality of covering**. Let $p : E \to B$ be any covering with B connected. Then p is trivial $\iff$ for every $e \in E$ there exists a section s of p such that $s(p(e)) = e$.

3. A Topological Space $X(\neq \phi)$ simply connected $\iff$ every covering $E \to X$ is trivial. Using this result, we shall also see that the circle S^1 is not simply connected as it has a non-trivial covering by $\mathbb{R}$. Infact, $\mathbb{R}$ is the universal cover of $\mathbb{S}^1$.

4. **The Monodromy action of the covering spaces.** Let B be any topological space with $p : E \to B$. If p is a covering, then the $\pi_1(B, b)$ acts on the fibre E_b. This action is called Monodromy. It measures how does a space behave when surrounded by a singularity. As a result, one can identify the category of covering spaces COV_b with the functor category $\mathrm{Fun}(\Pi_1(B), Set)$, where Π_1 is the fundamental groupoid.

5. If $p : E \to B$ is a locally trivial bundle, then there exists a Long exact sequence of pairs,

$$\ldots \pi_{n+1}(B, b) \to \pi_n(E_b, e) \to \pi_n(E, e) \to \pi_n(B, b) \ldots$$

6. Every covering gives rise to a special fibration called the *Serre Fibration.* A prototypical example is the fibration of spheres known as the *Hopf fibration* of $\mathbb{S}^3$ over $\mathbb{S}^2$ with fibres $\mathbb{S}^1$. It is locally a produt space but not a trivial bundle as globally $\mathbb{S}^3$ is not a product of $\mathbb{S}^2$ with $\mathbb{S}^1$. Thus, by combining with above fact we have the famous consequence from Homotopy theory that $\pi_n(\mathbb{S}^3) \cong \pi_n(\mathbb{S}^2)$, for $n \geq 3$.

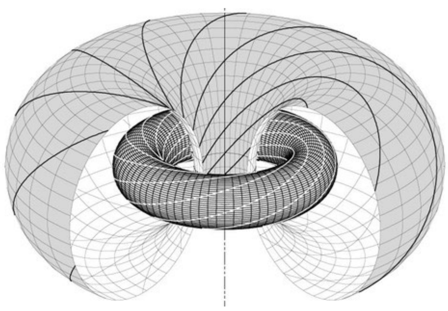


Figure 3: Hopf Fibration $\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2$

The other Hopf fibrations include:

- $\mathbb{S}^1$ over $\mathbb{S}^1$ with fibres $\mathbb{S}^0$
- $\mathbb{S}^7$ over $\mathbb{S}^4$ with fibres $\mathbb{S}^3$
- $\mathbb{S}^{15}$ over $\mathbb{S}^8$ with fibres $\mathbb{S}^7$

and that is all!

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